Journal of Approximation Theory **108**, 97–117 (2001)
doi:10.1006/jath.2000.3497, available online at http://www.idealibrary.com on IDEAL®

On Exact Values of n-Widths in a Hilbert Space[1]

Georgii G. Magaril-Il'yaev

Department of Mathematics, Moscow State Institute of Radio Engineering, Electronics and Automation (Technology University), pr. Vernadskogo 78, Moscow 117454, Russia

Konstantin Yu. Osipenko

Department of Mathematics, MATI–Russian State Technological University, Orshanskaya 3, Moscow 121552, Russia

and

Vladimir M. Tikhomirov

Department of Mechanics and Mathematics, Moscow State University, Vorobjovy Gory, Moscow 119899, Russia

Communicated by Allan Pinkus

Received February 22, 2000; accepted April 21, 2000;
published online November 28, 2000

The exact values of Kolmogorov n-widths have been calculated for two basic classes of functions. They are, on the one hand, classes of real functions defined by variation diminishing kernels and similar classes of analytic functions, and, on the other hand, classes of functions in a Hilbert space which are elliptical cylinders or generalized octahedra. This second case is surveyed and new results are presented. For n-widths of ellipsoids, elliptic cylinders, and generalized octahedra, upper bounds for the n-widths are based on the Fourier method. The lower bounds are based on the method of "embedded balls" for ellipsoids and the method of averaging for generalized octahedra. General theorems concerning elliptical cylinders and generalized octahedra are proved, various corollaries from these general theorems are considered, and some additional problems (average n-widths, extremal spaces for an ellipsoids and octahedra, etc.) are discussed. © 2001 Academic Press

[1] This research was supported in part by the Russian Foundation for Basic Research (Grants 96-01-10035 and 99-01-01181), the State Program on Support of Leading Scientific Schools (Grant 96-15-96072), and INTAS–OPEN 97-1050.

0021-9045/01 $35.00

1. INTRODUCTION

Let H be a Hilbert space and $C \subset H$ a centrally symmetric set. For $n \in \mathbb{Z}_+$ the *Kolmogorov n-widths* of C in H are given by

$$d_n(C, H) = \inf_{L_n} \sup_{x \in C} \inf_{y \in L_n} \|x - y\|_H,$$

where the left-most infimum is taken over all n-dimensional linear subspaces L_n of H (see [1–3]).

Let $r \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. Denote by $W_p^r(\mathbb{T})$ ($\mathbb{T} = [0, 2\pi)$) the Sobolev class of 2π-periodic functions whose $(r-1)$st derivatives are absolutely continuous and such that

$$\|x^{(r)}\|_{L_p(\mathbb{T})} := \left(\int_{\mathbb{T}} |x^{(r)}(t)|^p \, dt \right)^{1/p} \leqslant 1.$$

The subject considered in this paper goes back to two papers by Kolmogorov. In [3] Kolmogorov proved the formula

$$d_{2n-1}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = d_{2n}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = n^{-r}.$$

In the paper of Kolmogorov *et al.* [4], which was supplemented by Maltsev [5], the equality

$$d_n(Bl_1^N, l_2^N) = \sqrt{\frac{N-n}{N}}, \tag{1}$$

where

$$l_p^N := \left\{ x = (x_1, ..., x_N) \in \mathbb{R}^N \;\middle|\; \|x\|_{l_p^N}^p := \sum_{k=1}^{N} |x_k|^p \right\}, \qquad 1 \leqslant p < \infty,$$

and Bl_1^N is the unit ball in l_1^N was, in fact, proved. The authors of these papers did not actually state that they had calculated n-widths. (This was noted by Stechkin [6].) Note that the Sobolev class $W_2^r(\mathbb{T})$ is an elliptical cylinder (it is the orthogonal sum of the one-dimensional space of constants and a compact ellipsoid) and Bl_1^N is a regular octahedron in $\mathbb{R}^N$. We consider generalizations of these two results.

If $(X, \|\cdot\|)$ is a normed space, then we use the following notation:

$d(x, A, X) = \inf\{ \|x - y\| \mid y \in A\}$ is the distance from x to A in X.

$d(C, A, X) = \sup\{ d(x, A, X) \mid x \in C\}$ is the deviation of C from A in X.

2. n-WIDTHS OF ELLIPSOIDS AND ELLIPTICAL CYLINDERS

An *ellipsoid* is the image of a ball of a Hilbert space under a linear continuous mapping. If H and H_1 are Hilbert spaces, BH is the unit ball in H, and $T\colon H \to H_1$ is a linear continuous operator, then $T(BH)$ is an ellipsoid which we denote by $E(T)$. Let L be a finite-dimensional subspace in H_1. We call the orthogonal sum of an ellipsoid $E(T)$ and L,

$$E(T) \oplus L = \{ y = y_1 + y_2 \in H_1 \mid y_1 \in E(T),\ y_2 \in L,\ y_1 \perp y_2 \},$$

an *elliptical cylinder with base* $E(T)$ *and generalized axis* L.

Denote by N, $0 \leqslant N \leqslant \infty$ the dimension of span $E(T)$. We now calculate the n-widths of compact ellipsoids and elliptical cylinders with compact base.

Let T be a compact operator. By the Hilbert–Schmidt theorem (see, for example, [7, p. 231]) for the self-adjoint compact operator $T'T$ (T' is the adjoint operator to T) there exists an orthonormal system of eigenvectors $\{e_k\}_{k \geqslant 1}$ with corresponding eigenvalues s_k^2, $s_k \downarrow 0$, $s_k \neq 0$, such that each element $x \in H$ has the unique representation

$$x = \sum_{k \geqslant 1} \langle x, e_k \rangle e_k + \xi, \tag{2}$$

where $\xi \in \operatorname{Ker} T$. (The numbers s_k are called the s-numbers of T.)

The following theorem has been proved by many authors (see Section 5).

THEOREM 1 (n-Widths of Compact Ellipsoids). *Let H and H_1 be Hilbert spaces, let $T\colon H \to H_1$ be a compact operator, let $C = E(T) \oplus L_m$($\dim E(T) = N$, $\dim L_m = m$), and let $n \in \mathbb{Z}_+$. Then*

$$d_n(C, H_1) = \begin{cases} \infty, & n < m, \\ s_{n+1-m}, & m \leqslant n \leqslant N + m, \\ 0, & n > N + m. \end{cases}$$

The linear, Gel'fand, and Bernstein n-widths satisfy the same equalities.

Proof. We prove the theorem for the Kolmogorov n-width. The statement of the theorem for the cases where $n < m$ and $n > N + m$ may be easily checked. Let $n < N$ and for simplicity assume $m = 0$ (the general case easily follows from this). The upper bound will be proved by the *Fourier method.* Let $y \in E(T)$; that is, $y = Tx$, $\|x\|_H \leqslant 1$. By (2) $y = \sum_{k \geqslant 1} \langle x, e_k \rangle Te_k$ and $\|x\|_H^2 = \sum_{k \geqslant 1} |\langle x, e_k \rangle|^2 + \|\xi\|_H^2 \leqslant 1$. Let us approximate y by $S_n y = \sum_{k=1}^{n} \langle x, e_k \rangle Te_k$. Then taking into account the orthogonality of the system $\{Te_k\}$, we have

$$\| y - S_n y \|_{H_1}^2 = \sum_{k \geqslant n+1} s_k^2 \, |\langle x, e_k \rangle|^2 \leqslant s_{n+1}^2 \sum_{k \geqslant n+1} |\langle x, e_k \rangle|^2 \leqslant s_{n+1}^2 .$$

The upper bound is proved. The lower bound will be proved by the method of *embedded balls*. We consider the $(n+1)$-dimensional subspace $\hat{L}=\operatorname{span}\{Te_k\}_{k=1}^{n+1}$ of H_1 and show that the set $s_{n+1}BH_1\cap\hat{L}$ lies in $E(T)$. Let $y\in s_{n+1}BH_1\cap\hat{L}$. Then $y=\sum_{k=1}^{n+1}y_kTe_k$ and $\|y\|_{H_1}^2=\sum_{k=1}^{n+1}y_k^2s_k^2\leqslant s_{n+1}^2$. If $x=\sum_{k=1}^{n+1}y_ke_k$, then it is clear that $y=Tx$ and since

$$\|x\|_H^2=\sum_{k=1}^{n+1}y_k^2=\sum_{k=1}^{n+1}\frac{y_k^2s_k^2}{s_k^2}\leqslant\frac{1}{s_{n+1}}\sum_{k=1}^{n+1}y_k^2s_k^2\leqslant 1,$$

we obtain that $s_{n+1}BH_1\cap\hat{L}\subset E(T)$. By the theorem on n-widths of a ball (see, for example, [2, p. 12]), which is trivial for a Hilbert space, we have $d_n(E(T),H_1)\geqslant d_n(s_{n+1}BH_1\cap\hat{L},H_1)=s_{n+1}$. ∎

3. n-WIDTHS OF GENERALIZED OCTAHEDRA

In finite-dimensional geometry an octahedron is the convex hull of a simplex with a vertex at the origin and a simplex symmetric to it. For octahedra which are so defined there is no known general method for calculating the n-widths. But it is possible to calculate n-widths for octahedra in $\mathbb{R}^N$ which are the convex hulls of the vectors $\{\pm f_k\}$, $1\leqslant k\leqslant N$, obtained from one vector $K=(k_1,\dots,k_N)$ by cyclical permutation. Such octahedra may be considered Sobolev classes $W_1^K(\mathbb{Z}_N)$ consisting of functions $y=(y_1,\dots,y_N)$ on the cyclical group of order N defined by a convolution

$$W_1^K(\mathbb{Z}_N)=\{y\in\mathbb{R}^N\mid y=K*x,\ \|x\|_{l_1^N}\leqslant 1\},$$

where $x=(x_1,\dots,x_N)$, $y_i=\sum_{j=1}^{N}k_{i+j-1}x_j$, and the summation is carried out modulo N. The regular octahedra can be defined in this same way as a convolution class on the cyclical group $\mathbb{Z}_N$ with the kernel K equal to 1 at the zero of the group and 0 at all other elements.

This gives us the possibility of considering generalized Sobolev classes of so-called sourcewise represented functions which are similar to generalized octahedra.

First we recall some definitions (details may be found in [8]). Let G be a compact group with Haar measure μ ($\mu(G)=1$). Let $\{T^\alpha\}_{\alpha\in\mathscr{A}}$ (where $\mathscr{A}$ is at most a countable set) be a complete system of finite-dimensional nonreducible unitary representations of G. For each $\alpha\in\mathscr{A}$ we denote by $t_{ij}^\alpha(\cdot)$, $i,j=1,\dots,n_\alpha=\dim T^\alpha$, the matrix elements of the representation T^α in some orthonormal basis. These functions are continuous and the functions $\{\sqrt{n_\alpha}\,t_{ij}^\alpha(\cdot)\}$, $\alpha\in\mathscr{A}$, $i,j=1,\dots,n_\alpha$, form an orthonormal basis in $L_2(G)$. Note that if G is an Abelian group, then all representations T^α, $\alpha\in\mathscr{A}$, are one-dimensional. For each $\alpha\in\mathscr{A}$ and $1\leqslant j\leqslant n_\alpha$ set $H_j^\alpha=$

$\operatorname{span}\{t^{\alpha}_{ij}(\cdot) \mid i=1, ..., n_\alpha\}$. The space $L_2(G)$ is represented as the direct sum of those spaces which are left-translation-invariant.

A set X is called a *homogeneous* G*-space* if the group G acts transitively on X—in other words, if there exists a map $G \times X \to X$, $(g, x) \to gx$, such that $(g_2 g_1)\, x = g_2(g_1 x)$, $ex = x$ (e is the unity element of G) for all $g_1, g_2 \in G$ and $x \in X$, and in addition for every $x_1, x_2 \in X$ there exists a $g \in G$ for which $x_2 = gx_1$. It is obvious that any group G is a homogeneous G-space with respect to the operation $(g, g_0) \to gg_0$.

Let $x_0 \in X$, let $H = \{g \in G \mid gx_0 = x_0\}$, and let G/H be the set of (left) residue classes of group G on the subgroup H. Consider the map $p: X \to G/H$ which associates x with the residue class gH such that $gx_0 = x$. The map p is a one-to-one mapping. Thus any function on X may be considered as a function on G which is constant on the residue classes. By virtue of this fact, for any topological homogeneous G-space X with compact group G and measure ν invariant with respect to G (that is, $\nu(A) = \nu(gA)$ for any measurable subset $A \subset X$ and $g \in G$), the structure of $L_2(X)$ is analogous to the structure of $L_2(G)$. More precisely $L_2(X)$ is a direct sum of at most an enumerable set of finite-dimensional subspaces H_k invariant with respect to G consisting of continuous functions.

We will need the following auxiliary result.

LEMMA 1. *Let X be a topological homogeneous G-space with compact group G and probability measure invariant with respect to G. If $\{e_k(\cdot)\}_{k=1}^{n}$ is an orthonormal system of continuous functions from $L_2(X)$ such that $L_n = \operatorname{span}\{e_k(\cdot)\}_{k=1}^{n}$ is invariant with respect to G, then*

$$\sum_{k=1}^{n} |e_k(\cdot)|^2 \equiv n.$$

Proof. For $x \in X$ consider the function

$$\xi_x(\cdot) = \sum_{k=1}^{n} \overline{e_k(x)}\, e_k(\cdot).$$

If $y(\cdot) \in L_n$ then it is clear that

$$\langle y(\cdot), \xi_x(\cdot) \rangle = y(x). \tag{3}$$

Let $x_1, x_2 \in X$ and $g \in G$ such that

$$x_2 = gx_1. \tag{4}$$

Using the invariance of the measure, (3), and (4), for every $y(\cdot)\in L_n$ we have

$$\langle y(\cdot), \xi_{x_2}(g\,\cdot)\rangle = \langle y(g^{-1}\cdot), \xi_{x_2}(\cdot)\rangle = y(g^{-1}x_2) = y(x_1) = \langle y(\cdot), \xi_{x_1}(\cdot)\rangle.$$

Thus $\xi_{x_2}(g\,\cdot)=\xi_{x_1}(\cdot)$. Substituting here x_1 we obtain that

$$\sum_{k=1}^{n} |e_k(x_1)|^2 = \sum_{k=1}^{n} |e_k(x_2)|^2 = C.$$

Since the given measure is a probability measure and the system $\{e_k(\cdot)\}_{k=1}^{n}$ is orthonormal we have

$$C=\int_X \sum_{k=1}^{n} |e_k(x)|^2\, d\mu(x) = n. \quad \blacksquare$$

Let X be a topological homogeneous G-space with compact group G and probability measure invariant with respect to G. As was mentioned, in this case $L_2(X)$ may be represented in the form

$$L_2(X) = \sum_{k\geqslant 1} H_k, \qquad \dim H_k = n_k < \infty,$$

where the H_k are shift-invariant spaces of continuous functions. Consider the classes of functions represented as convolutions with kernels

$$K(t,\tau) = \sum_{k\geqslant 1} \sum_{j=1}^{n_k} \gamma_{kj} e_{kj}(t)\, \overline{e_{kj}(\tau)}, \tag{5}$$

where the $\{e_{kj}(\cdot)\}_{j=1}^{n_k}$ is an orthonormal basis for H_k of continuous functions and $\gamma_{kj}\in\mathbb{C}$ are such that

$$\sum_{k\geqslant 1} n_k \max_{1\leqslant j\leqslant n_k} |\gamma_{kj}|^2 < \infty. \tag{6}$$

The function $K(\cdot,\cdot)$ induces the operator

$$Ax(t) = \int_X K(t,\tau)\, x(\tau)\, d\mu(\tau).$$

We show that A is a continuous operator from $L_1(X)$ into $L_2(X)$. Indeed, by the Cauchy–Bunyakovsky inequality for all $t\in X$

$$|Ax(t)| \leqslant \int_X (|K(t,\tau)|^2\, |x(\tau)|)^{1/2}\, |x(\tau)|^{1/2}\, d\mu(\tau)$$

$$\leqslant \left(\int_X |K(t,\tau)|^2\, |x(\tau)|\, d\mu(\tau)\right)^{1/2} \|x(\cdot)\|^{1/2}_{L_1(X)}.$$

Squaring this inequality, integrating it, and then changing the order of integration, we obtain

$$\|Ax(\cdot)\|^2_{L_2(X)} \leqslant \|x(\cdot)\|^2_{L_1(X)} \sup_{t\in X} \int_X |K(t,\tau)|^2\, d\mu(\tau)$$

$$= \|x(\cdot)\|^2_{L_1(X)} \sup_{t\in X} \|K(t,\cdot)\|^2_{L_2(X)}.$$

According to the Parseval equality for all $t \in X$

$$\|K(t,\cdot)\|^2_{L_2(X)} = \sum_{k\geqslant 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2\, |e_{kj}(t)|^2.$$

By Lemma 1 $\sum_{j=1}^{n_k} |e_{kj}(t)|^2 = n_k$. In view of (6) we have that $\|K(t,\cdot)\|_{L_2(X)} \leqslant C$. Therefore A: $L_1(X) \to L_2(X)$ is a continuous operator.

Set

$$W_1^K(X) = \{\, y(\cdot) \in L_2(X) \mid y(\cdot) = Ax(\cdot),\ \|x(\cdot)\|_{L_1(X)} \leqslant 1\}.$$

THEOREM 2. *Let X be a topological homogeneous G-space with compact group G and probability measure invariant with respect to G. Let K: $X \times X \to \mathbb{C}$ be a function of the form* (5), *where the γ_{kj} satisfy the additional condition $|\gamma_{kj}| = \lambda_k$, $1 \leqslant j \leqslant n_k$, $k \geqslant 1$. Assume that $\{\lambda_k\}_{k\geqslant 1}$ are in decreasing order. Then for all $n = n_1 + \cdots + n_s$*

$$d_n(W_1^K(X), L_2(X)) = \left(\sum_{k\geqslant s+1} \lambda_k^2 n_k\right)^{1/2}.$$

Proof. Since $W_1^K(X) = \mathrm{cl\,co}\{K(\cdot,\tau)\}_{\tau\in X}$ it is sufficient to prove the statement of the theorem for the set $\{K(\cdot,\tau)\}_{\tau\in X}$.

The Upper Bound. We use the Fourier method to project our class onto the subspace $L_n = \mathrm{span}\{e_{kj}(\cdot) \mid 1 \leqslant j \leqslant n_k,\ 1 \leqslant k \leqslant s\}$. Then for any $\tau \in X$ using the Parseval equality, the hypothesis of the theorem, and Lemma 1, we have

$$d^2(K(\cdot,\tau), L_n, L_2(X)) = \sum_{k\geqslant s+1} \lambda_k^2 n_k.$$

Hence the required estimate follows.

The Lower Bound. We use the method of averaging. Let L_n be an n-dimensional subspace of $L_2(X)$, and $\{f_m\}_{m=1}^n$ an orthonormal basis of L_n. Then for all $\tau \in X$

$$d^2(K(\cdot,\tau), L_n, L_2(X)) = \|K(\cdot,\tau)\|^2_{L_2(X)} - \sum_{m=1}^{n} \left| \int_X K(t,\tau)\, \overline{f_m(t)}\, d\mu(t) \right|^2. \tag{7}$$

In view of the hypothesis of the theorem and by Lemma 1 we have

$$\begin{aligned} \|K(\cdot,\tau)\|^2_{L_2(X)} &= \sum_{k\geqslant 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2\, |e_{kj}(t)|^2 \\ &= \sum_{k\geqslant 1} \lambda_k^2 \sum_{j=1}^{n_k} |e_{kj}(t)|^2 = \sum_{k\geqslant 1} \lambda_k^2 n_k. \end{aligned} \tag{8}$$

Furthermore,

$$\left| \int_X K(t,\tau)\, \overline{f_m(t)}\, d\mu(t) \right|^2 = \left| \sum_{k\geqslant 1} \sum_{j=1}^{n_k} \gamma_{kj} \overline{e_{kj}(\tau)} \int_X e_{kj}(t)\, \overline{f_m(t)}\, d\mu(t) \right|^2.$$

Substituting it and (8) into (7), integrating the obtained expression, and using the Parseval equality with the hypothesis of the theorem, we obtain

$$\begin{aligned} &\int_X d^2(K(\cdot,\tau), L_n, L_2(X))\, d\mu(\tau) \\ &\quad = \sum_{k\geqslant 1} \lambda_k^2 n_k - \sum_{m=1}^{n} \sum_{k\geqslant 1} \sum_{j=1}^{n_k} |\gamma_{kj}|^2 \left| \int_X e_{kj}(t)\, \overline{f_m(t)}\, d\mu(t) \right|^2 \\ &\quad = \sum_{k\geqslant 1} \lambda_k^2 n_k - \sum_{m=1}^{n} \sum_{k\geqslant 1} \lambda_k^2 \sum_{j=1}^{n_k} \left| \int_X e_{kj}(t)\, \overline{f_m(t)}\, d\mu(t) \right|^2. \end{aligned} \tag{9}$$

For $k \geqslant 1$ set

$$\hat{c}_k = \sum_{m=1}^{n} \sum_{j=1}^{n_k} \left| \int_X e_{kj}(t)\, \overline{f_m(t)}\, d\mu(t) \right|^2.$$

It is easy to check that $0 \leqslant \hat{c}_k \leqslant n_k$ and $\sum_{k\geqslant 1} \hat{c}_k = n_1 + \cdots + n_s$. Consider the problem of linear programming

$$\sum_{k\geqslant 1} \lambda_k^2 c_k \to \max, \quad 0 \leqslant c_k \leqslant n_k, \qquad \sum_{k\geqslant 1} c_k = n_1 + \cdots + n_s.$$

The solution of this problem is evidently

$$c_k = n_k, \quad 1 \leqslant k \leqslant s, \qquad c_k = 0, \quad k \geqslant s+1$$

(we recall that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$). Thus we obtain the lower bound for the left-hand side of (9)

$$c_k = n_k, \quad 1 \leqslant k \leqslant s, \qquad c_k = 0, \quad k \geqslant s+1.$$

Standard arguments now lead to the required estimate. ▮

4. COROLLARIES FROM THE GENERAL THEOREMS

We begin with n-widths of convolution classes of functions defined on a compact group. Let G be a compact group and let $K(\cdot) \in L_2(G)$. The operator of convolution is defined as

$$Tx(g) = \int_G K(gs^{-1})\, x(s)\, d\mu(s). \tag{10}$$

It is a compact operator from $L_2(G)$ into $L_2(G)$. Moreover, it follows from the Minkowski inequality that (10) is a continuous operator from $L_1(G)$ into $L_2(G)$. Set

$$W_p^K(G) = \{\, y(\cdot) \in L_2(G) \mid y(\cdot) = Tx(\cdot),\ \|x(\cdot)\|_{L_p(G)} \leqslant 1\}, \qquad p = 1, 2.$$

THEOREM 3. *Let G be a compact group and let $K(\cdot) \in L_2(G)$ be such that its Fourier coefficients c_{ij}^{α} when expanded in the orthonormal basis $e_{ij}^{\alpha}(\cdot) = \sqrt{n_\alpha}\, t_{ij}^{\alpha}(\cdot)$, $\alpha \in \mathscr{A}$, $i, j = 1, \ldots, n_\alpha$, satisfy the condition: for any $\alpha \in \mathscr{A}$ the matrix $C_\alpha = (c_{ij}^{\alpha})_{i,j=1}^{n_\alpha}$ has the form $\lambda_\alpha U_\alpha$, where $\lambda_\alpha \in \mathbb{C}$ and U_α is a unitary matrix. Assume that $\{\lambda_k/\sqrt{n_k}\}_{k \geqslant 1}$ is the sequence $\{\lambda_\alpha/\sqrt{n_\alpha}\}_{\alpha \in \mathscr{A}}$ ordered in decreasing order. Then for all $m \geqslant 1$ $(n_0 = 0)$*

(1) *for any n such that $n_1^2 + \cdots + n_{m-1}^2 < n \leqslant n_1^2 + \cdots + n_m^2$*

$$d_n(W_2^K(G), L_2(G)) = \frac{|\lambda_m|}{\sqrt{n_m}},$$

(2) *for any $n = n_1^2 + \cdots + n_{m-1}^2 + n_m s_m$, where $1 \leqslant s_m \leqslant n_m$,*

$$d_n(W_1^K(G), L_2(G)) = \left(|\lambda_m|^2 (n_m - s_m) + \sum_{k \geqslant m+1} |\lambda_k|^2 n_k \right)^{1/2}.$$

Proof. (1) Using properties of the matrix elements

$$t_{ij}^{\alpha}(g_1 g_2) = \sum_{k=1}^{n_\alpha} t_{ik}^{\alpha}(g_1)\, t_{kj}^{\alpha}(g_2), \qquad t_{ij}^{\alpha}(g) = \overline{t_{ji}^{\alpha}(g^{-1})},$$

we have

$$K(gs^{-1}) = \sum_{\alpha \in \mathscr{A}} \sum_{i,\, j=1}^{n_\alpha} c_{ij}^{\alpha} e_{ij}^{\alpha}(gs^{-1}) = \sum_{\alpha \in \mathscr{A}} \sum_{i,\, j=1}^{n_\alpha} \frac{1}{\sqrt{n_\alpha}}\, c_{ij}^{\alpha} \sum_{k=1}^{n_\alpha} e_{ik}^{\alpha}(g)\, \overline{e_{jk}^{\alpha}(s)}. \tag{11}$$

It is easy to verify that if $x = (x_1, ..., x_{n_\alpha})$ is an eigenvector of $C'_\alpha C_\alpha$ for the eigenvalue λ, then for all $1 \leqslant k \leqslant n_\alpha$ the functions $\sum_{j=1}^{n_\alpha} x_j e_{jk}^{\alpha}(\cdot)$ are eigenfunctions for $T'T$ with eigenvalue $\lambda/\sqrt{n_\alpha}$. Consequently, s_j are the s-numbers of $T'T$. It remains to use Theorem 1.

(2) By (11) we have

$$K(gs^{-1}) = \sum_{\alpha \in \mathscr{A}} \frac{1}{\sqrt{n_\alpha}} \sum_{k,\, i=1}^{n_\alpha} e_{ik}^{\alpha}(g) \sum_{j=1}^{n_\alpha} c_{ij}^{\alpha} \overline{e_{jk}^{\alpha}(s)}.$$

Set

$$E_k^{\alpha}(g) := \begin{pmatrix} e_{1k}^{\alpha}(g) \\ \vdots \\ e_{n_\alpha k}^{\alpha}(g) \end{pmatrix}.$$

Then

$$\begin{aligned} K(gs^{-1}) &= \sum_{\alpha \in A} \frac{1}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} (E_k^{\alpha}(g), \bar{C}_\alpha E_k^{\alpha}(s))_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} (E_k^{\alpha}(g), \bar{U}_\alpha E_k^{\alpha}(s))_\alpha, \end{aligned}$$

where

$$(a, b)_\alpha := \sum_{i=1}^{n_\alpha} a_i \bar{b}_i.$$

There exists a unitary matrix V_α such that

$$V_\alpha U_\alpha V_\alpha^* = \begin{pmatrix} \gamma_1^{\alpha} & & 0 \\ & \ddots & \\ 0 & & \gamma_{n_\alpha}^{\alpha} \end{pmatrix} =: \Gamma_\alpha,$$

where $|\gamma_j^\alpha| = 1$, $j = 1, ..., n_\alpha$. Set

$$F_k^\alpha(g) = \bar{V}_\alpha E_k^\alpha(g) =: \begin{pmatrix} f_{1k}^\alpha(g) \\ \vdots \\ f_{n_\alpha k}^\alpha(g) \end{pmatrix}.$$

Then $E_k^\alpha(g) = \bar{V}_\alpha^* F_k^\alpha(g)$, $f_{ik}^\alpha(g)$ is an orthonormal basis, and

$$\begin{aligned} K(gs^{-1}) &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} (\bar{V}_\alpha^* F_k^\alpha(g), \bar{U}_\alpha \bar{V}_\alpha^* F_k^\alpha(s))_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} (F_k^\alpha(g), \bar{V}_\alpha \bar{U}_\alpha \bar{V}_\alpha^* F_k^\alpha(s))_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k=1}^{n_\alpha} (F_k^\alpha(g), \bar{\Gamma}_\alpha F_k^\alpha(s))_\alpha \\ &= \sum_{\alpha \in A} \frac{\lambda_\alpha}{\sqrt{n_\alpha}} \sum_{k,\, j=1}^{n_\alpha} \gamma_j^\alpha f_{jk}^\alpha(g) \overline{f_{jk}^\alpha(s)}. \end{aligned}$$

Using Theorem 2 we obtain the desired equality. ∎

COROLLARY 1. *Let G be a compact Abelian group, let $K(\cdot) \in L_2(G)$, and let c_j, $j \geqslant 1$, be Fourier coefficients of K in an orthonormal basis formed by characters of the group. Assume that the c_j are arranged in decreasing order. Then for all $n \in \mathbb{Z}_+$*

$$d_n(W_2^K(G), L_2(G)) = |c_{n+1}|,$$

$$d_n(W_1^K(G), L_2(G)) = \left(\sum_{j \geqslant n+1} |c_j|^2 \right)^{1/2}.$$

COROLLARY 2. *For $K = (k_1, ..., k_N)$ set*

$$c_j = \sum_{m=1}^{N} k_m e^{-2\pi i (j-1) m/N}.$$

Assume that c_j are arranged in decreasing order. Then for all $n \in \mathbb{Z}_+$

$$d_n(W_2^K(\mathbb{Z}_N), L_2(\mathbb{Z}_N)) = |c_{n+1}|,$$

$$d_n(W_1^K(\mathbb{Z}_N), L_2(\mathbb{Z}_N)) = \left(\frac{1}{N} \sum_{j \geqslant n+1} |c_j|^2 \right)^{1/2}.$$

If $K=(1,0,...,0)$, then from the last equality we obtain (1).

Let

$$\mathbb{S}^d=\left\{x=(x_1,...,x_{d+1})\in\mathbb{R}^{d+1}\;\middle|\;\sum_{j=1}^{d+1}x_j^2=1\right\}$$

be the unit sphere. It is known (see [9]) that $L_2(\mathbb{S}^d)=\sum_{k=0}^{\infty}H_k$, where

$$\dim H_k=n_k=\binom{d+k}{k}-\binom{d+k-1}{k-2}$$

(H_k is the set of spherical harmonics of order k). Let $\{Y_j^k\}_{j=1}^{n_k}$ be an orthonormal basis of H_k. For the Laplace operator Δ and any $x(\cdot)\in H_k$ the equality

$$\Delta x(\cdot)=-\lambda_k x(\cdot)$$

holds where $\lambda_k=k(k+d-1)$. For $\alpha>0$ the operator $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}x(\cdot)=\sum_{k=1}^{\infty}\lambda_k^{\alpha/2}\sum_{j=1}^{n_k}x_{kj}Y_j^k(\cdot),$$

where $x(\cdot)\in L_2(\mathbb{S}^d)$ and $x(\cdot)=\sum_{k=0}^{\infty}\sum_{j=1}^{n_k}x_{kj}Y_j^k(\cdot)$.

Set

$$W_2^\alpha(\mathbb{S}^d)=\{x(\cdot)\in L_2(\mathbb{S}^d)\mid \|(-\Delta)^{\alpha/2}x(\cdot)\|_{L_2(\mathbb{S}^d)}\leqslant 1\}.$$

It is easy to check that this class can be represented in the form

$$W_2^\alpha(\mathbb{S}^d)=\{x(\cdot)\in L_2(\mathbb{S}^d)\mid x(\cdot)=c+Ty(\cdot),\ c\in\mathbb{R},\ \|y(\cdot)\|_{L_2(\mathbb{S}^d)}\leqslant 1,\ y(\cdot)\perp 1\},$$

where for $y(\cdot)=\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}y_{kj}Y_j^k(\cdot)$

$$Ty(\cdot)=\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}\lambda_k^{-\alpha/2}y_{kj}Y_j^k(\cdot).$$

COROLLARY 3. *Let* $n_0+\cdots+n_{k-1}\leqslant n<n_0+\cdots+n_k$. *Then*

$$d_n(W_2^\alpha(\mathbb{S}^d),L_2(\mathbb{S}^d))=\lambda_k^{-\alpha/2}.$$

The class $W_2^\alpha(\mathbb{S}^d)$ for $d=1$ and $\alpha=r\in\mathbb{Z}_+$ coincides with the Sobolev class

$$W_2^r(\mathbb{T})=\{x(\cdot)\in L_2(\mathbb{T})\mid x^{(r-1)}(\cdot)\text{ abs. cont., }\|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})}\leqslant 1\}.$$

In this case $\lambda_k=k^2$, $n_0=1$, $n_k=2$, $k\geqslant 1$. Thus we obtain

Corollary 4. *For all* $n \in \mathbb{Z}_+$

$$d_{2n-1}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = d_{2n}(W_2^r(\mathbb{T}), L_2(\mathbb{T})) = \frac{1}{n^r}.$$

One does not obtain results similar to those obtained in Corollary 3 and 4 for the classes $W_1^{\alpha}(\mathbb{S}^d)$ and $W_1^r(\mathbb{T})$. The reason for this is the additional condition $y(\cdot) \perp 1$ which does not permit us to apply Theorem 2. Some estimates of $d_n(W_1^r(\mathbb{T}), L_2(\mathbb{T}))$ may be found in [2, p. 101].

5. AVERAGE WIDTHS

In this section we calculate exact values of average Kolmogorov widths for some classes of functions defined on $\mathbb{R}^d$ and $\mathbb{Z}^d$ in the L_2 metric. We begin with the definition of the average dimension of a subspace. Let $G = \mathbb{R}^d$ or $\mathbb{Z}^d$ and μ_G be the Lebesgue measure on G if $G = \mathbb{R}^d$, and discrete measure if $G = \mathbb{Z}^d$. Let $\mathscr{A}(G)$ be the set of positive numbers if $G = \mathbb{R}^d$ and the set of natural numbers if $G = \mathbb{Z}^d$. Assume that A is a subset of $L_p(G)$ $(1 \leqslant p \leqslant \infty)$ and $\alpha \in \mathscr{A}(G)$. Denote by A_α the restriction of A to the set

$$\square_\alpha = \{t = (t_1, ..., t_d) \in G \mid |t_j| \leqslant \alpha, j = 1, ..., d\}.$$

Let L be a subspace of $L_p(G)$. For every $\varepsilon > 0$ and $\alpha \in \mathscr{A}(G)$ consider the value

$$K_\varepsilon(\alpha, L, L_p(G)) = \min\{n \in \mathbb{Z}_+ \mid d_n((L \cap BL_p(G))_\alpha, L_p(\square_\alpha)) < \varepsilon\},$$

where $BL_p(G)$ is the unit ball of $L_p(G)$. It is clear that $K_\varepsilon(\alpha, L, L_p(G))$ is the minimal dimension of a subspace of $L_p(\square_\alpha)$ which approximates the set $(L \cap BL_p(G))_\alpha$ to within ε. It is easy to check that for every $\varepsilon > 0$ the function $\alpha \to K_\varepsilon(\alpha, L, L_p(G))$ does not decrease, and obviously for every $\alpha > 0$ the function $\varepsilon \to K_\varepsilon(\alpha, L, L_p(G))$ does not increase.

The *average dimension of* L *in* $L_p(G)$ is defined as

$$\overline{\dim}(L, L_p(G)) = \lim_{\varepsilon \to 0} \liminf_{\alpha \to \infty} \frac{K_\varepsilon(\alpha, L, L_p(G))}{\mu_G(\square_\alpha)}.$$

It is clear that $\overline{\dim}(L, L_p(\mathbb{Z}^d)) \leqslant 1$.

We shall formulate here one result about average dimension of functional spaces which we later need. Let G^* be $\mathbb{R}^d$ for $G = \mathbb{R}^d$ and $\mathbb{T}^d$ (a d-dimensional torus) for $G = \mathbb{Z}^d$. Let μ_{G^*} be the Lebesgue measure on G^* divided by $(2\pi)^d$. (The last condition is connected with the fact that μ_G is the natural Haar measure on G as on a locally compact Abelian group and μ_{G^*} is just the Haar measure associated with it on the dual group G^*.)

Let A be a subset of G^* and $1 \leqslant p \leqslant \infty$. Set

$$\mathscr{B}_{A,p}(G) = \{x(\cdot) \in L_p(G) \mid \operatorname{supp} \hat{x}(\cdot) \subset A\},$$

where $\operatorname{supp} \hat{x}(\cdot)$ is the support of Fourier transform of $x(\cdot)$ ($x(\cdot)$ is considered as a distribution). It is clear that $\mathscr{B}_{A,p}(G)$ is a subspace of $L_p(G)$.

Recall that a set $A \subset G^*$ is called Jordan measurable if its characteristic function is integrable in the sense of Riemann. The following two theorems (Theorems 4 and 5) were proved in [10, 11] for $G = \mathbb{R}^d$. In the general case these theorems may be proved in much the same way.

THEOREM 4 (About Average Dimension). *Let $G = \mathbb{R}^d$ or $\mathbb{Z}^d$, let A be a Jordan measurable subset of G^* with finite measure, and let $1 \leqslant p \leqslant \infty$. Then*

$$\overline{\dim}(\mathscr{B}_{A,p}(G), L_p(G)) = \mu_{G^*}(A).$$

The notion of average dimension leads at once to the corresponding analogue of Kolmogorov n-width. Let C be a centrally symmetric subset of $L_p(G)$ and $v \geqslant 0$. The *Kolmogorov average v-width of C in $L_p(G)$* is defined as

$$\bar{d}_v(C, L_p(G)) = \inf_{L} \sup_{x(\cdot) \in C} \inf_{y(\cdot) \in L} \|x(\cdot) - y(\cdot)\|_{L_p(G)},$$

where the first infimum is taken over all subspaces L of $L_p(G)$ such that $\overline{\dim}(L, L_p(G)) \leqslant v$. Any subspace for which this infimum is attained we call an *extremal subspace for $\bar{d}_v(C, L_p(G))$.*

The following analogue of the theorem holds about widths of a ball for average widths.

THEOREM 5 (About Widths of a Ball). *Let $A \subset G^*$ be a Jordan measurable set, let $v > 0$, and let $\mu_{G^*}(A) > v$. Then*

$$\bar{d}_v(\mathscr{B}_{A,p}(G) \cap BL_p(G), L_p(G)) = 1.$$

We calculate the exact value of the average width in L_2 for the set C which is a convolution class of functions defined on G. If $K(\cdot) \in L_2(G)$, then the operator of convolution with this kernel $x(\cdot) \to K * x(\cdot)$ is obviously a linear continuous operator from $L_2(G)$ into $L_2(G)$. Set

$$W_2^K(G) = \{ y(\cdot) \in L_2(G) \mid y(\cdot) = (K * x)(\cdot),\ \|x(\cdot)\|_{L_2}(G) \leqslant 1\}.$$

Denote by $\hat{z}(\cdot)$ the Fourier transform of the function $z(\cdot) \in L_2(G)$.

THEOREM 6. *Let* $K(\cdot)\in L_1(G)\cap L_2(G)$, *let* $v>0$ *if* $G=\mathbb{R}^d$, *and let* $0<v<1$ *if* $G=\mathbb{Z}^d$. *Then*

$$\bar{d}_v(W_2^K(G), L_2(G))=\hat{K}^*(v),$$

where $\hat{K}^*(\cdot)$ *is the non-decreasing permutation of* $\hat{K}(\cdot)$. *Moreover, the space* $\mathscr{B}_{A(v),2}(G)$, *where* $A(v)=\{\tau\in G^*\,|\,|\hat{K}(\tau)|>\hat{K}^*(v)\}$, *is an extremal space for* $\bar{d}_v(W_2^K(G), L_2(G))$.

Proof: The Lower Bound. We use the method of "embedded balls." For every $\gamma>0$ consider the set $A(\gamma)=\{\tau\in G^*\,|\hat{K}(\tau)|>\gamma\}$. Since $K(\cdot)\in L_1(G)$ the function $\hat{K}(\cdot)$ is continuous and $\hat{K}(\tau)\to 0$ as $|\tau|\to\infty$ if $G=\mathbb{R}^d$. Therefore the set $A(\gamma)$ is Jordan measurable. We prove that

$$\mathscr{B}_{A(\gamma),2}(G)\cap\gamma BL_2(G)\subset W_2^K(G). \tag{12}$$

Indeed, let $y(\cdot)$ belong to the left-hand side of (12). Assume that $x(\cdot)\in L_2(G)$ is defined by the condition: $\hat{x}(\tau)=0$ almost everywhere if $\tau\notin A(\gamma)$ and $\hat{x}(\tau)=\hat{K}^{-1}(\tau)\,\hat{y}(\tau)$ almost everywhere if $\tau\in A(\gamma)$. Thus $y(\cdot)=(K*x)(\cdot)$ and we have to show that $\|x(\cdot)\|_{L_2(G)}\leqslant 1$. By the Plancherel theorem

$$\begin{aligned}\|x(\cdot)\|^2_{L_2(G)}&=\int_G|\hat{K}^{-1}(\tau)\,\hat{y}(\tau)|^2\,d\tau=\int_{A(\gamma)}|\hat{K}^{-1}(\tau)\,\hat{y}(\tau)|^2\,d\tau\\&\leqslant\gamma^{-2}\int_{A(\gamma)}|\hat{y}(\tau)|^2\,d\tau\leqslant\gamma^{-2}\|y(\cdot)\|^2_{L^2(G)}\leqslant 1;\end{aligned}$$

that is, $y(\cdot)\in W_2^K(G)$.

Now $\gamma>0$ is such that $\mu_{G^*}(A(\gamma))>v$. By Theorem 4

$$\overline{\dim}(\mathscr{B}_{A(\gamma),2}(G), L_2(G))=\mu_{G^*}(A(\gamma))>v.$$

Then by Theorem 5 (taking into account the obvious property of homogeneity of these widths) we obtain

$$\bar{d}_v(\mathscr{B}_{A(\gamma),2}(G)\cap\gamma BL_2(G), L_2(G))=\gamma.$$

From this and (12), using the monotonicity of widths it follows that

$$\bar{d}_v(W_2^K(G), L_2(G))>\gamma.$$

Passing to the supremum in this inequality over all $\gamma>0$ for which $\mu_{G^*}(A(\gamma))>v$ (in view of the continuity of $\hat{K}(\cdot)$ this is equivalent to passage to the supremum over all $\gamma>0$ for which $\mu_{G^*}(A(\gamma))\geqslant v$) we obtain the required lower bound.

The Upper Bound. This is based on the approximation of $W_2^K(G)$ by the Fourier method. Let $\gamma=\gamma(v)$ be such that $\mu_{G^*}(A(\gamma))=v$ (it is clear that in this case $\gamma=\hat{K}^*(v)$). By Theorem 4 $\overline{\dim}(\mathscr{B}_{A(\gamma),2}(G), L_2(G))=v$. With every $y(\cdot)=(K*x)(\cdot)\in W_2^K(G)$ associate the function $\eta(\cdot)\in\mathscr{B}_{A(\gamma),2}(G)$ such that $\hat{\eta}(\cdot)=\chi_{A(\gamma)}(\cdot)\,\hat{y}(\cdot)$, where $\chi_{A(\gamma)}(\cdot)$ is the characteristic function of the set $A(\gamma)$. By using the Plancherel theorem and the definition of $W_2^K(G)$ we estimate $\|y(\cdot)-\eta(\cdot)\|_{L_2(G)}$ and obtain the required upper bound. ∎

The next result is the analogue of (1) for average widths.

THEOREM 7. *Let* $0<v<1$. *Then*

$$\bar{d}_v(Bl_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d))=\sqrt{1-v}.$$

Proof. The arguments do not depend on the dimension d so for simplicity we consider the case $d=1$.

The Lower Bound. Let L be a subspace of $l_2(\mathbb{Z})$, let $\overline{\dim}(L, l_2(\mathbb{Z}))\leqslant v$, and let $\varepsilon>0$. Assume that $\{N_k\}$ is a subsequence of natural numbers such that

$$\liminf_{N\to\infty}\frac{K_\varepsilon(N, L, l_2(\mathbb{Z}))}{2N+1}=\lim_{k\to\infty}\frac{K_\varepsilon(N_k, L, l_2(\mathbb{Z}))}{2N_k+1}. \tag{13}$$

By the definition of average dimension, for every k there exists a subspace $M_k\subset l_2^{2N_k+1}$ such that

$$d((L\cap Bl_2(\mathbb{Z}))_k, M_k, l_2^{2N_k+1})<\varepsilon, \tag{14}$$

$$\dim M_k\leqslant K_\varepsilon(N_k, L, l_2(\mathbb{Z})). \tag{15}$$

Let $x\in Bl_1^{2N_k+1}$. Extending x by zero to $\mathbb{Z}$ we obtain $x\in Bl_1(\mathbb{Z})$ and consequently $x\in Bl_2(\mathbb{Z})$. Let $y\in L$ and $z\in M_k$ be such that

$$\|y-z\|_{l_2^{2N_k+1}}=d(y, M_k, l_2^{2N_k+1}). \tag{16}$$

Then using the triangle inequality, (16), (14), and again the triangle inequality, we have

$$\begin{aligned}\|x-y\|_{l_2(\mathbb{Z})}&\geqslant\|x-y\|_{l_2^{2N_k+1}}\geqslant\|x-z\|_{l_2^{2N_k+1}}-\|y-z\|_{l_2^{2N_k+1}}\\&\geqslant d(x, M_k, l_2^{2N_k+1})-d(y, M_k, l_2^{2N_k+1})\\&\geqslant d(x, M_k, l_2^{2N_k+1})-\varepsilon\,\|y\|_{l_2(\mathbb{Z})}\\&\geqslant d(x, M_k, l_2^{2N_k+1})-\varepsilon\,\|x-y\|_{l_2(\mathbb{Z})}-\varepsilon\,\|x\|_{l_2(\mathbb{Z})}.\end{aligned}$$

Consequently,

$$(1+\varepsilon)\,\|x-y\|_{l_2(\mathbb{Z})} \geqslant d(x, M_k, l_2^{2N_k+1}) - \varepsilon.$$

Hence,

$$(1+\varepsilon)\, d(Bl_1(\mathbb{Z}), L, l_2(\mathbb{Z})) \geqslant d(Bl_1^{2N_k+1}, M_k, l_2^{2N_k+1}) - \varepsilon. \tag{17}$$

From (13) (taking into account (15)) it follows that for every $0<\delta<1-\nu$ there exists k_0 such that for all $k \geqslant k_0$

$$\dim M_k \leqslant K_\varepsilon(N_k, L, l_2(\mathbb{Z})) \leqslant (\nu+\delta)(2N_k+1).$$

Put $N(k)=2N_k+1$ and $n(k)=[(\nu+\delta)(2N_k+1)]$. Then $\dim M_k \leqslant n(k) < N(k)$. Taking into account these inequalities, (17), and (1), we have

$$\begin{aligned}(1+\varepsilon)\, d(Bl_1(\mathbb{Z}), L, l_2(\mathbb{Z}) &\geqslant d_{n(k)}(Bl_1^{N(k)}, l_2^{N(k)}) - \varepsilon\\ &= \sqrt{1-\frac{n(k)}{N(k)}} - \varepsilon \geqslant \sqrt{1-(\nu+\delta)} - \varepsilon.\end{aligned}$$

In view of the arbitrariness of ε, δ, and L we obtain the required lower bound.

The Upper Bound. Let $\varepsilon>0$ and let the numbers $n, N \in \mathbb{N}$ be chosen so that $n<N$ and $(n/N) \leqslant \nu \leqslant (n/N)+\varepsilon$. Denote by $L_{n,N}$ a subspace of l_2^N with dimension at most n which is extremal for $d_n(Bl_1^N, l_2^N)$. We consider this subspace as a subspace of functions on $\mathbb{Z}$ with support on $\{0, 1, ..., N-1\}$. Let $e_i(\cdot)$, $i=1, ..., N$, be a basis for $L_{n,N}$. If $k \in \mathbb{Z}$, then the functions $e_i(\cdot+kN)$, $i=1, ..., N$, form a basis in the space of all functions from $L_{n,N}$ shifted by kN. Denote by L the set of functions $y(\cdot)$ defined on $\mathbb{Z}$ which have the form $y(\cdot)=\sum_{k\in\mathbb{Z}}\sum_{i=1}^n x_{ki} e(\cdot+kN)$, where $\sum_{k\in\mathbb{Z}}\sum_{i=1}^n x_{ki}^2 < \infty$. It is clear that L is a subspace of $l_2(\mathbb{Z})$.

We show that $\overline{\dim}(L, l_2(\mathbb{Z})) \leqslant \nu$. Indeed, denote by L_m the restriction of L to $\{-mN, ..., mN\}$. It is easy to see that $\dim L_m \leqslant 2mn+1$ and therefore

$$\overline{\dim}(L, l_2(\mathbb{Z})) \leqslant \liminf_{m\to\infty} \frac{2mn+1}{2mN+1} = \frac{n}{N} \leqslant \nu.$$

Denote by M_k the restriction of L to $\Delta_k=\{kN, ..., (k+1)\,N-1\}$.

Now let $x \in Bl_1(\mathbb{Z})$ and let x_k be the restriction of x to Δ_k. Since $x_k \in \|x_k\|_{l_1^N} Bl_1^N$ and $M_k = L_{n,N}$ (if $L_{n,N}$ is considered as a set of functions defined on Δ_k), there exists $y_k \in M_k$ for which

$$\|x_k - y_k\|_{l_2^N} \leqslant \sqrt{1-(n/N)}\ \|x_k\|_{l_1^N}. \tag{18}$$

Let $y \in L$ be a function such that the restriction of y to Δ_k equals y_k. Then using (18) and the mean inequality we have

$$\|x-y\|_{l_2(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} \|x_k - y_k\|^2_{l_2^N}\right)^{1/2} \leqslant \sqrt{1-\frac{n}{N}}\left(\sum_{k\in\mathbb{Z}} \|x_k\|^2_{l_1^N}\right)^{1/2}$$
$$\leqslant \sqrt{1-\frac{n}{N}} \sum_{k\in\mathbb{Z}} \|x_k\|_{l_1^N} = \sqrt{1-\frac{n}{N}}\, \|x\|_{l_1(\mathbb{Z})}$$
$$\leqslant \sqrt{1-\frac{n}{N}} \leqslant \sqrt{1-\nu+\varepsilon}.$$

In view of the arbitrariness of ε we obtain the required estimate. ∎

We note here one general fact which in particular enables us to obtain at once a series of extremal spaces for the widths $d_n(Bl_1^N, l_2^N)$ and $\bar{d}_\nu(Bl_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d))$.

Let G be a locally compact Abelian group (LCAG), let G^* be the dual group to G (that is, the group of all continuous characters on G), and let $\operatorname{ch}(g, g^*)$ be the value of $g^* \in G^*$ at the element $g \in G$. We define by μ_G (μ_{G^*}) the Haar measure on G (G^*).

For every $x(\cdot) \in L_1(G)$ the function $\hat{x}(\cdot)$ defined on G^* which is given by the formula

$$\hat{x}(g^*) = \int_G x(g) \operatorname{ch}(-g, g^*)\, d\mu_G \tag{19}$$

is called the *Fourier transform of* $x(\cdot)$. By (19) it follows that $\hat{x}(\cdot)$ is a continuous function and

$$\|\hat{x}(\cdot)\|_{C(G^*)} \leqslant \|x(\cdot)\|_{L_1(G)}. \tag{20}$$

The Fourier transform can be extended up to an isometric operator from $L_2(G)$ onto $L_2(G^*)$ (this extension we define by the same symbol $\hat{x}(\cdot)$). Thus we have the Parseval equality

$$\|x(\cdot)\|_{L_2(G)} = \|\hat{x}(\cdot)\|_{L_2(G^*)}. \tag{21}$$

If G is a discrete group, then the dual group G^* is compact and we shall usually assume that $\mu_{G^*}(G^*) = 1$.

Let A be a nonempty subset of G^* and $p = 1$ or 2. Set

$$\mathscr{B}_{A,p}(G) = \{x(\cdot) \in L_p(G) \mid \operatorname{supp} \hat{x}(\cdot) \subset A\},$$

where $\operatorname{supp} \hat{x}(\cdot)$ is the support of $\hat{x}(\cdot)$. It is clear that $\mathscr{B}_{A,p}(G)$ is a subspace of $L_p(G)$.

PROPOSITION 1. *Let G be a discrete LCAG and let A be a measurable subset of G^*. Then $L_1(G)$ is embedded in $L_2(G)$ and*

$$d(BL_1(G), \mathscr{B}_{A,2}(G), L_2(G)) \leqslant \sqrt{1-\mu_{G^*}(A)}.$$

Proof. Let $x(\cdot) \in BL_1(G)$ and let the function $y(\cdot) \in L_2(G)$ be such that $\hat{y}(\cdot) = \chi_A(\cdot)\,\hat{x}(\cdot)$ ($\chi_A(\cdot)$ is the characteristic function of A). It is clear that $y(\cdot) \in \mathscr{B}_{A,2}(G)$. Using (21) and (20), we have

$$\|x(\cdot)-y(\cdot)\|^2_{L_2(G)} = \int_{G^*\setminus A} |\hat{x}(g^*)|^2\, d\mu_{G^*} \leqslant \|\hat{x}(\cdot)\|^2_{C(G^*)} \int_{G^*\setminus A} d\mu_{G^*}$$
$$\leqslant \|x(\cdot)\|^2_{L_1(G)}\,(1-\mu_{G^*}(A)) \leqslant 1-\mu_{G^*}(A).$$

If we take here $x(\cdot) \in L_1(G)$ and $y(\cdot) = 0$, then we obtain that $\|x(\cdot)\|_{L_2(G)} \leqslant \|x(\cdot)\|_{L_1(G)}$. This means that $L_1(G)$ is continuously embedded in $L_2(G)$. ▮

We apply this result to the problems mentioned above.

1. The space l_p^N, $1 \leqslant p \leqslant \infty$, can be considered as $L_p(G)$, where $G = \mathbb{Z}_N = \{0, 1, ..., N-1\}$ is a finite discrete Abelian group with the operation of addition modulo N. Characters of this group are the functions $k \to \exp(2\pi kl/N)$, $k \in \mathbb{Z}_N$, where $0 \leqslant l \leqslant N-1$. Therefore we can identify the dual group $\mathbb{Z}_N^*$ with $\mathbb{Z}_N$. Let $n < N$ and $A = \{l_{j_1}, ..., l_{j_n}\} \subset \mathbb{Z}_N^*$. It is clear that $\mu_{\mathbb{Z}_N^*}(A) = n/N$. Consider the space $L_n = \operatorname{span}\{\exp(2\pi i l_{j_1} \cdot /N), ..., \exp(2\pi i l_{j_n} \cdot /N)\}$, $\dim L_n = n$. From Proposition 1 and (1) it follows that L_n is an extremal subspace for $d_n(Bl_1^N, l_2^N)$.

2. Let $A \subset \mathbb{T}^d$ be Jordan measurable, $\mu_{\mathbb{T}^*}(A) = v$, $0 < v < 1$. Consider the space $L_v = \{x(\cdot) \in l_2(\mathbb{Z}^d) \mid \operatorname{supp} \hat{x}(\cdot) \subset A\}$. By Theorem 4 we have $\overline{\dim}(L_v, l_2(\mathbb{Z}^d)) = v$. Now from Proposition 1 and Theorem 4 it follows that L_v is an extremal subspace for $\bar{d}_v(Bl_1(\mathbb{Z}^d), l_2(\mathbb{Z}^d))$.

6. COMMENTS

Various statements which are equivalent to Theorem 1 were proved by many authors (see [2, 12–15]). Of course this result was known to Kolmogorov who considered in [3] only particular cases of elliptical cylinders.

In a finite-dimensional space n-widths of regular octahedra were in fact obtained in two papers, [4] (the upper bound) and [5] (the lower bound). It is interesting to note that Kolmogorov in 1948 did not take into consideration that in these papers $d_n(Bl_1^N, l_2^N)$ were calculated. This fact was noted by Stechkin [6], who used it to find asymptotic values of n-widths for functional classes.

There is one more type of octahedra for which it is possible to calculate exact values of widths. They are octahedra with different axes

$$Bl_1^N(a) := \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{k=1}^{N} \frac{|x_k|}{a_k} \leqslant 1 \right\}, \qquad a_1 \geqslant \cdots \geqslant a_N.$$

For the dual case Smolyak [16] found the exact values of the linear (λ_n) and Gel'fand (d^n) n-widths,

$$\lambda_n(Bl_2^N(a), l_\infty^N) = d^n(Bl_2^N(a), l_\infty^N) = \max_{m>n} \sqrt{\frac{m-n}{\sum_{k=1}^m a_k^{-2}}}.$$

For the Kolmogorov n-width $d_n(Bl_1(a), l_2)$ the exact result was obtained by Sofman [17, 18] (see also [19]).

In the continuous case estimates for the n-widths of generalized octahedra and even more general sets (images of compacts under continuous transformation in the Hilbert space) can be obtained using results such as a theorem of Ismagilov [20] which is based on the method of averages (we demonstrated this method in the proof of Theorem 2). Ismagilov cited Obukhov [21] as a predecessor in using the method of averages. Several statements of a similar type which are used to calculate exact values of n-widths for classes of analytic functions can be found in [22–24]. In those papers the dual situation is considered and the exact values of linear, Gel'fand, and Bernstein widths of $W_2^K(X)$ in $C(X)$ are found. In the dual case using the Hilbert space structure it is possible to calculate the exact values of n-widths for $W_2^\alpha(\mathbb{S}^d)$ and $W_2^r(\mathbb{T})$.

The concept of average dimension takes its beginnings from the definition of "average entropy" for stochastic signals with bounded spectrum, which was given by Shannon [25]. Kolmogorov further modified this definition for determined functions. Then Tikhomirov [26] defined the notion of average dimension replacing entropy by Kolmogorov n-widths. The definition of the average dimension used in this paper is a modification of Tikhomirov's definition. The notion of the Kolmogorov average widths is due to Magaril-Il'yaev.

REFERENCES

1. V. M. Tikhomirov, "Approximation Theory," Encyclopedia of Math. Sci., Vol. 14 [Analysis, II], Springer-Verlag, Berlin, 1990.
2. A. Pinkus, "n-Widths in Approximation Theory," Springer-Verlag, Berlin, 1985.
3. A. N. Kolmogorov, Über die beste Annäherung von Functionen einer gegebenen Functionenklasse, *Ann. of Math.* **37** (1936), 107–110.
4. A. N. Kolmogorov, A. A. Petrov, and Yu. M. Smirnov, A formula of Gauss in the theory of the method of least squares, *Izv. Akad. Nauk SSSR Ser. Mat.* **11** (1947), 561–566.

5. A. I. Maltsev, A remark on the paper of A. N. Kolmogorov, A. A. Petrov, and Yu. M. Smirnov "A Formula of Gauss on the Theory of Least Squares," *Izv. Akad. Nauk SSSR Ser. Mat.* **11** (1947), 567–568.
6. S. B. Stechkin, On the best approximation of given classes by any polynomials, *Uspekhi Mat. Nauk* **9** (1954), 133–134.
7. A. N. Kolmogorov and S. V. Fomin, "Elements of Theory of Functions and Functional Analysis," Nauka, Moscow, 1972.
8. A. Weil, "L'intégration dans les Groupes Topologiques et ses Applications," Hermann, Paris, 1965.
9. E. M. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
10. G. G. Magaril-Il'yaev, Mean dimension, widths, and optimal recovery of Sobolev classes of functions on the line, *Mat. Sb.* **182** (1991), 1635–1656; English translation *Math. USSR Sb.* **74**, 381–403.
11. G. G. Magaril-Il'yaev, Average widths of Sobolev classes on $\mathbb{R}^n$, *J. Approx. Theory* **76** (1994), 65–76.
12. D. T. Allahverdiev, On the rate of approximation of completely continuous operators by finite-dimensional operators, *Azerb. Gos. Univ. Uchen. Zap.* **2** (1957), 27–35.
13. V. M. Tikhomirov, Diameters of sets in functional spaces and the theory of best approximations, *Uspekhi Mat. Nauk* **15** (1960), 81–120; English translation *Russian Math. Surveys* **15**, 75–111.
14. V. M. Tikhomirov, "Some Questions of Approximation Theory," Moscow State University, Moscow, 1976.
15. A. Pietsch, "Operator Ideals," North-Holland, Amsterdam, 1980.
16. S. A. Smolyak, "On Optimal Restoration of Functions and Functionals of Them," Candidate dissertation, Moscow State University, Moscow, 1965.
17. L. B. Sofman, Diameters of octahedra, *Mat. Zametki* **5** (1969), 429–436; English translation *Math. Notes* **5**, 258–262.
18. L. B. Sofman, Diameters of an infinite-dimensional octahedra, *Vestn. Mosk. Univ. Mat.* **28** (1973), 54–56; English translation *Moscow Univ. Math. Bull* **28**, 45–47.
19. C. V. Hutton, J. S. Morrel, and J. R. Retherford, Diagonal operators, approximation numbers, and Kolmogoroff diameters, *J. Approx. Theory* **16** (1976), 48–80.
20. R. S. Ismagilov, On n-dimensional diameters in a Hilbert space, *Funktsional. Anal. i Prilozhen.* **2** (1968), 32–39; English translation *Funct. Anal. Appl.* **2**, 125–132.
21. A. M. Obukhov, On statistically orthogonal decomposition of empirical function, *Izv. Akad. Nauk SSSR Ser. Geophys.* **3** (1960), 432–439.
22. K. Yu. Osipenko and M. I. Stessin, On n-widths of the Hardy class H^2 in the unit ball of $\mathbb{C}^n$, *Uspekhi Mat. Nauk* **45** (1990), 193–194; English translation *Russian Math. Surveys* **45**, 235–236.
23. K. Yu. Osipenko, On N-widths of holomorphic functions of several variables, *J. Approx. Theory* **82** (1995), 135–155.
24. K. Yu. Osipenko and O. G. Parfenov, Ismagilov type theorems for linear, Gel'fand and Bernstein n-widths, *J. Complexity* **11** (1995), 474–492.
25. C. E. Shannon, A mathematical theory of communication, *Bell System Techn. J.* **27** (1948), 379–423, 623–656.
26. V. M. Tikhomirov, On approximation characteristics of multivariate smooth functions, *in* "Proceedings of the Conference on Differential Equations and Applied Mathematics," pp. 183–188, Nauka, SO AN SSSR, 1980.

Printed in Belgium